ON ORDINARY POINTS IN ARRANGEMENTS

BY

WALTER MEYER

ABSTRACT

We show that in an arrangement of n curves in the plane (or on the sphere) there are at least $n/2$ points where precisely 2 curves cross (ordinary points). Furthermore there are at least $(4/3)n$ triangular regions in the complex determined by the arrangement. Triangular regions and ordinary vertices are both connected with boundary vertices of certain distinguished subcomplexes. By analogy with rectilinear planar polygons we distinguish concave and convex vertices of these subcomplexes. Our lower bounds arise from lower bounds for convex vertices in the distinguished subcomplexes.

1. Introduction

A well-known problem of Sylvester states: given n points ($n \geqq 3$) in the plane, not all on a line, must there exist ordinary lines, that is, lines which contain precisely two of the given points. The best answer to date is [2]: there are always at least $3n/7$ ordinary lines. In dual form, where one starts with an arrangement of lines and asks for ordinary points (that is, points incident with only two lines) the problem suggests two natural generalizations: we may consider families of pseudolines (a *pseudoline* is the image of a line under a homeomorphism of the plane onto itself) or families of simple closed curves. The lower bound $3n/7$ has recently been established for arrangements of pseudolines [3]. The purpose of this note is to establish the lower bound $n/2$ for digon-free arrangements of curves. For convenience we will formulate the problem on the two-dimensional sphere instead of in the plane.

An *arrangement* of curves on the sphere is a finite family of simple closed curves on the sphere such that each pair of curves have exactly two points in common and actually cross at these intersection points. These intersection points

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are called vertices or points of the arrangement. A vertex of an arrangement is called *ordinary* if it lies on exactly two curves of the arrangement. An arrangement of curves is called *trivial* if there are precisely two vertices. All arrangements will henceforth be assumed non-trivial. (Actually, requiring an arrangement of curves to contain no *digons* that is, regions with just two incident vertices also rules out the trivial arrangement of curves).

We state the following theorems immediately and outline their proofs in Section 3.

THEOREM 1. *In a digon-free arrangement on n curves, $n \geqq 3$, there are at least $n/2$ ordinary vertices.*

THEOREM 2. *In a digon-free arrangement of n curves there are at least $(4/3)n$ triangles.*

2. Definitions and notation

An arrangement can be thought of as a graph on the sphere whose vertices are the vertices as previously defined and where an edge is a minimal arc lying in a curve of the arrangement having a pair of vertices as endpoints. In the usual way this graph determines regions which are closures of connected components of the complement of the graph. These vertices, edges and faces are the 0-cells, 1-cells and 2-cells of a complex on the sphere

If G is a curve of the arrangement, G divides the sphere into two connected components. We call the closure of such a component a combinatorial half-plane. If H is such a combinatorial half-plane corresponding to the curve G, we define a complex $I(G,H)$ as follows: the vertices of $I(G,H)$ are those of $H \setminus G$; the edges and faces of $I(G,H)$ are those of H which have the further property that all their incident vertices are in $H \setminus G$.

If e is an edge of the arrangement and has x as one endpoint then $x(e)$ is that edge of the arrangement, distinct from e, having x as one endpoint, and lying on the same curve as e.

Let K be a subcomplex of $I(G,H)$. Suppose x is a vertex in the boundary of $|K|$, the union of the cells of K. We call x a convex corner of K if the edges at x, but lying in K, are consecutive in the clockwise or counterclockwise order around x (equivalently, the edges at x but not in K are consecutive around x) and for each edge e at x in K, $x(e) \notin K$. We call x concave if the edges at x lying in K are consecutive around x and if there exist two edges e and e', at x and lying in K, where

$x(e)$ and $x(e') \in K$ but $x(e) \neq e'$. Note that not all vertices in the boundary of $|K|$ are corners (see Figure 1).

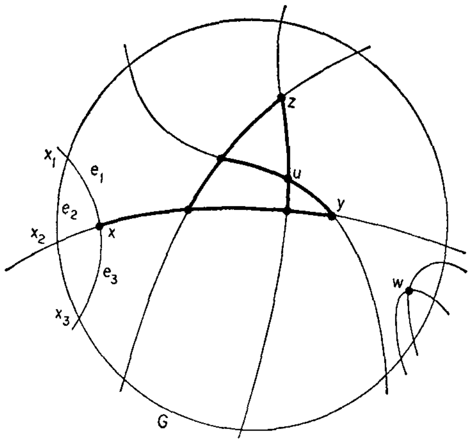

Fig. 1

$I(G,H)$ is shown in heavy line. $I(G,H)$ has the following convex corners: w, x, y, z. $I(G,H)$ has u as a concave corner. The tepees $T(x,G)$, $T(z,G)$, $T(w,G)$ are minimal but $T(y,G)$ is not.

Now let x be a convex corner of K and let $e_1, e_2, \cdots, e_k$ be the consecutive edges (in the order of their subscripts) at x not in K. Let e_i have endpoints x and x_i. If $K = I(G,H)$, then $x_i \in G$ and we define an arc on G containing these x_i as follows. The direction around x for which the e_i are consecutive in the order of their subscripts induces a direction on G. If we traverse an arc from x_1 to x_n on G in this direction we encounter the x_i in the order of their subscripts. (We may, however, encounter other vertices of the arrangement; Figure 1 shows an example). Call this arc γ.

There is a simple closed curve Γ formed by xx_1, γ, x_kx and it divides the sphere into two complementary domains, one of which lies in H. The complex consisting of Γ and the vertices, edges, and faces in the aforementioned domain is called a tepee and denoted $T(x, x_1, \cdots, x_k)$ or, sometimes for brevity, $T(x,G)$. (Henceforth we shall avoid circumlocutions when dealing with complexes in a half-plane H by using words such as "interior" or "simply-connected" with the understanding that we mean relative to the space H rather than the whole sphere.) The vertex x is the *apex* of the tepee and x_1 and x_k are end vertices of the base or simply *end vertices*. It should be noted that the notation $T(x,G)$ is ambiguous in the case where $\{x\}$ is a component of K because there are, in this case, as many tepees with apex x as there are edges at x. Such isolated vertices will need somewhat special handling anyhow so there should be no confusion about the notation.

3. Outline of proofs of Theorems 1 and 2

Tepees and convex vertices are useful for the study of ordinary vertices because of the following lemmas. The proof of the first, which is somewhat lengthy and tedious, we defer until Section 4 so as not to obscure the essentially simple lines of the proofs of the theorems.

LEMMA 3. *If K is a simply-connected complex contained in $I(G,H)$ then K contains at least two corners convex in K unless K is a single vertex.*

COROLLARY 4. *Each component of $I(G,H)$ contains at least two vertices which are convex in that component and hence in $I(G,H)$, unless that component consists of a single vertex.*

LEMMA 5. *If x and y are distinct convex corners in the same component of $I(G,H)$ then the two tepees $T(x,G)$ and $T(y,G)$ intersect in at most a pair of end vertices on G.*

PROOF. Let Γ_x be the bounding circuit of $T(x,G)$, composed of edges xx_1, and xx_k and an arc γ_x joining x_1 and x_k on G. Let Γ_y be the bounding circuit of $T(y,G)$, composed of edges yy_1, yy_t and an arc γ_y joining y_1 and y_t on G. If the tepees intersect, then Γ_x and Γ_y intersect. They cannot do so along the edges xx_i or yy_j so $\gamma_x \cap \gamma_y \neq 0$. However, if γ_x and γ_y intersect in more than a pair of vertices, then $\gamma_x \subset \gamma_y$ or $\gamma_y \subset \gamma_x$. Suppose the latter case for example. Then $T(y,G) \subset T(x,G)$. Now there cannot be any path in the graph of $I(G,H)$ connecting x to y, for if there were, this would contradict the consecutivity condition on the convex corner x.

LEMMA 6. *For any curve G and associated combinatorial half-plane H, $I(G,H)$ contains at least two tepees with distinct apexes which are minimal with respect to containment and are of the form $T(x,G)$, unless $I(G,H)$ consists of a single vertex.*

PROOF. $I(G,H)$ is not empty, for if it were H would contain digons.

In the special case where $I(G,H)$ consists of two isolated vertices the lemma holds because at each of these vertices, among the various tepees which exist there, there is precisely one tepee not containing the other vertex, and hence minimal. The more general case where $I(G,H)$ consists of k isolated vertices is an easy induction. When an isolated vertex x and its incident edges is added, at most one minimal tepee in the previous complex can be ruined, but exactly one new one is created at x. Consequently, unless $I(G,H)$ consists of a single isolated vertex, we can assume that $I(G,H)$ has a component K with more than one vertex. Then Corollary 4 states that K possesses two corners, x and y, which are convex in K (and hence in $I(G,H)$). Since the associated tepees $T(x,G)$ and $T(y,G)$ intersect in at most two vertices, according to Lemma 5, then the minimal tepees contained in $T(x,G)$ and $T(y,G)$ respectively are distinct.

LEMMA 7. *Let* $T(x,x_1,x_2,\cdots,x_p)$ *be a tepee.*

(i) *If* $p = 2$, *then* x *is ordinary.*

If the tepee is minimal then

(ii) *the circuits formed by* xx_i, x_ix_{i+1}, $x_{i+1}x$ $(i = 1,2,\cdots,p-1)$ *each bound a single triangular cell of the base complex, and*

(iii) *if* $p > 2$ *then* $x_2,x_3,\cdots,x_{p-1}$ *are ordinary.*

PROOF. (i) For every curve through x there are one or more edges xx_i contained in that curve. To establish part (i), it is only necessary to show that two such edges cannot both terminate at the same x_i. This fact, which will be used again later, is a case of the result: two vertices, x and y, of a digon-free arrangement of curves are joined by at most one edge. To establish this, let $e_1, e_2,\cdots,e_n$ be an enumeration of the edges at x in consecutive order. If e_i and e_j both terminate at y and $i<j$ then e_{i+1} also terminates at y. Now e_i and e_{i+1} must also be consecutive around y or there would be a curve passing through y twice, which is impossible. Edges e_i and e_{i+1} are also consecutive around y. Thus e_i and e_{i+1} enclose a digon, which is a contradiction. Therefore part (i) is established.

(ii) If the circuit Γ_i formed by xx_i, x_ix_{i+1}, $x_{i+1}x$ does not bound a single region which is a triangle, then $I(G,H)$ is disconnected and contains a component interior to Γ_i, say A. If A is an isolated vertex, clearly we can choose a tepee from A contained in $T(x,x_1,\cdots,x_p)$, contradicting the minimality. Otherwise let w_1 and w_2 be two convex vertices of A. It is easy to see that at least one of the tepees $T(w_i,G)$ is contained in Γ_i. This contradicts the minimality of $T(x,x_1,\cdots,x_p)$.

(iii) This is immediate from (ii) and the fact that curves of an arrangement cross each other where they meet.

We now begin the proof of Theorem 1. The determination of the bound for m, the number of ordinary vertices, proceeds as follows. (The author wishes to thank

E. Bender for a suggestion improving an earlier counting argument.) Let k be the number of distinct triples (x, G, H) which have these properties:

(a) H is a combinatorial half-plane determined by G;

(b) $x \in H$;

(c) x is ordinary and either lies on G or is the apex of a minimal tepee with base on G.

COROLLARY 8. *For each pair* (G, H) *satisfying condition* (a) *there are at least two vertices* x *satisfying conditions* (b) *and* (c).

PROOF. By Lemma 6, each of the half-planes H determined by G either has two minimal tepees with distinct apexes or has a single vertex not on G. In the former case, Lemma 5 and Lemma 7 (i) and (iii) give the result. Now consider the latter case, where $I(G, H)$ is a single vertex x. If $e_1, e_2, \cdots, e_k$ $(k \geqq 4)$ are the edges at x then let the other endpoints of these e_i be x_i. Each $x_i \in G$. Since curves cross where they meet and the arrangement contains no digons, each x_i is ordinary. Thus, in this case, we have at least four ordinary vertices in H.

Now there are $2n$ possible pairs (G, H) where H is a half-plane determined by G. For each of these pairs there are, according to Corollary 8, at least two vertices satisfying (a) and (b). Thus $k \geqq 4n$.

Now consider a particular ordinary vertex x. There are precisely two curves containing x and they provide four triples (x, G, H) satisfying conditions (a), (b), and (c). Additional triples involving x and satisfying these conditions may arise when x is the apex of minimal tepees to curves G_j, each G_j passing through two or more consecutive x_i. There can be at most four such curves G_j for if there were more, then two consecutive points x_i and x_{i+1} would be adjacent along two different curves, say G_1 and G_2. Then x_i and x_{i+1} are joined by two edges, and as we saw in the proof of Lemma 7 (i), this cannot happen in a digon-free arrangement of curves. Thus there are at most four curves G_j. (Note that there could be as few as one in the case where $x = I(G, H)$ for some G and H.) This implies that for a given ordinary vertex x there are not more than eight triples (x, G, H) satisfying conditions (a), (b), and (c). Thus $k \leqq 8m$. Combining the last two inequalities, we arrive at the result of the theorem: $m \geqq n/2$. This concludes the proof of Theorem 1 (except for Lemma 3 whose proof is in Section 4). We also can conclude the following theorem.

THEOREM 9. *In a digon-free arrangement of* n *curves there are at least* $4/3\,n$ *triangles.*

PROOF. If $T(x, G)$ is a minimal tepee, Lemma 7 (ii) shows that it contains a triangle incident with G. By Lemma 6 there are at least four minimal tepees of the form $T(x, G)$ for a fixed G, so there are at least four triangles incident with G.

Now let k be the number of pairs (T, G) where T is a triangle with an edge on the curve G. If p_3 is the number of triangles in the arrangement, $k = 3p_3$. But each curve is incident with at least four triangles, so $k \geqq 4n$ whence $3p_3 \geqq 4n$ and $p_3 \geqq 4n/3$.

It should be noted that Grünbaum [1] has made two conjectures concerning the results of Theorems 1 and 2.

Conjecture I. A digon-free arrangement of n curves has at least $n-1$ ordinary vertices. Furthermore, if it has exactly $n-1$ then $n \equiv 1 \pmod 3$.

Conjecture II. A digon-free arrangement of curves has at least $2n-4$ triangles.

4. Convex corners

In this section we intend to prove Lemma 3. Let K be a complex which satisfies the hypotheses of Lemma 3, but which has less than two convex corners and which has the minimal total number of cells (of all dimensions), say t, among all complexes with these properties. We proceed by deducing properties that such a complex K would have to possess and then by discovering that these properties imply the existence of two convex corners, which is a contradiction. The first stage is to show that K is a simple circuit with interior. Thereafter we focus attention on the pattern of convex, concave and other vertices on the bounding circuit of K, and on the curves which form these boundary vertices. The permissible patterns turn out to be fairly restrictive and always include two convex vertices.

We begin the first stage by making several observations.

Observation I. K is connected. For if K has K_1 and K_2 as components, by the minimality of K, each of K_1 and K_2 either has two convex corners or consists of a single vertex. In any combination of these cases, K will have at least two convex corners, a contradiction.

Observation II. The 1-skeleton of K (the graph consisting of all vertices and edges of K) is 2-connected, that is, it cannot be disconnected by removal of a single vertex. For otherwise K could be written as $K_1 \cup K_2$ where $K_1 \cap K_2$ is the single vertex x. By the minimality of K, each of K_1 and K_2 has two convex corners, at most one of which can be x. Consequently, K has at least two convex corners, a contradiction.

Observation III. There exists a simple circuit Γ where $K = \Gamma \cup \operatorname{int} \Gamma$. To establish this, we first note that K does not consist of a single edge with its end vertices, because the latter would both be convex. Now the 1-skeleton of K contains a circuit, for otherwise it would not be 2-connected unless it consisted of a single edge, the case just ruled out. Let Γ be a simple circuit in the 1-skeleton of K which is maximal with respect to the number of cells of K in $\Gamma \cup \operatorname{int} \Gamma$. Since K is connected, either every cell of K lies in $\Gamma \cup \operatorname{int} \Gamma$ or there is an edge $(x, y) \in K$ with $x \in \Gamma$, $y \notin \Gamma \cup \operatorname{int} \Gamma$. In the latter case let z be a vertex of Γ different from x. Since the 1-skeleton of K is 2-connected, there is a path from y to z missing x. From this we can extract a simple subpath missing int Γ from y to some point of Γ other than x. Let P denote this path with (x, y) appended at the beginning. Out of Γ and P we can construct a simple circuit which encloses more cells than Γ, contradicting the maximality of Γ. Thus each cell of K lies in $\Gamma \cup \operatorname{int} \Gamma$. Since K is simply connected, $K = \Gamma \cup \operatorname{int} \Gamma$.

Now the corners of K clearly lie on Γ. In preparation for the study of the patterns of these corners, we need to make some definitions. Two corners x and y are called *adjacent* provided there is an arc γ contained in Γ with x and y as endpoints and such that γ contains no other corners of K (it may contain vertices on the boundary Γ which are not corners). Such an arc connecting two adjacent corners is called a *geodesic* arc. Clearly a geodesic arc is contained entirely in some curve of the arrangement.

Suppose K has a concave corner x and suppose a and b are either edges or geodesic arcs of Γ which meet at x. For any edge or a geodesic arc e, denote by $G(e)$ that curve of the arrangement on which e lies. Define $A(x, a)$ to be the arc contained in $G(a)$ with x as one endpoint and the other endpoint, denoted $w(x, a)$, being the first vertex of Γ encountered as one traverses $G(a)$, beginning at x, in the direction from x which avoids the edge a. $A(x, a)$ is said to *surface* (or to be a surfacing arc) if $w(x, a)$ belongs to the maximal geodesic arc containing b; otherwise $A(x, a)$ is said to *dive* (or to be a diving arc). $A(x, b)$ and the notions of surfacing and diving for this arc are defined similarly by interchanging the roles of a and b. Figure 2 illustrates these notions.

Let y be another concave corner of K connected to the adjacent concave corner x by the geodesic arc γ. Let c and d be the edges of Γ at y, the notation so

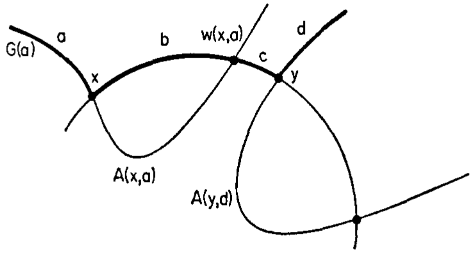

Fig. 2

$A\ (x,a)$ is a surfacing arc, $A\ (y,d)$ is a diving arc, Γ (shown in part).

chosen that $b, c \in \gamma$. Then we may assert our fourth and final observation.

Observation IV. For adjacent concave vertices x and y and the rest of the notation as above, $A(x, a)$ and $A(y, d)$ cannot both dive. For if they did, we could remove a complex from K and contradict the minimality as follows. Let $H(a)$ and $H'(a)$ be the half-planes determined by the curve $G(a)$, $H(a)$ being the one which contains b; let $H(d)$ and $H'(d)$ be the half-planes determined by the curve $G(d)$, $H(d)$ being the one which contains c. In case $A(x, a)$ and $A(y, d)$ intersect once, let $R = K \cap H(a) \cap H(d)$ and then define K' to consist of all cells lying in the closure of $K \setminus R$. Since K was assumed minimal, K' has at least two convex corners. None of these convex corners lies along $A(x, a)$ or $A(y, d)$ except possibly at $w(x, a)$ in the case when $w(x, a) = w(y, d)$ and in this case, $w(x, a)$ is convex in K also. In any case, restoring R leaves at least two convex corners in K, a contradiction. In case $A(x, a)$ and $A(y, d)$ are disjoint or intersect twice, consider the complexes $H'(a) \cap K$ and $H'(d) \cap K$. Each of these has two convex corners and again none can be along $A(x, a)$ or $A(y, d)$ except possibly at $w(x, a)$ or $w(y, d)$. Thus each complex has at least one convex corner other than $w(x, a)$ and $w(y, d)$ and these will be convex corners in K also. These corners cannot be identical since $|A(x, a) \cap A(y, d)| = 0$ or 2 whence these corners are on different parts of the arc Γ. Thus J has at least two convex corners, a contradiction.

We can describe the structure of K around the boundary Γ with fair precision for complexes K which satisfy the hypotheses of Lemma 3, have less than two convex corners, are minimal for these properties with respect to the total number of cells, and which consequently satisfy observations I, II, and III. In fact, we can describe the structure of K with enough precision so that we can show that K has at least two convex corners, contrary to its definition.

LEMMA 10. *If x is a concave vertex of K and a and b are edges of Γ touching x then at least one of $A(x,a)$ and $A(x,b)$ dives.*

PROOF. If both surface, then the curves $G(a)$ and $G(b)$ intersect at least three times.

The concave corner x is said to be of *dual type* if one of $A(x,a)$ and $A(x,b)$ dives and the other surfaces.

LEMMA 11. *There exists a concave corner of dual type in K.*

PROOF. First note that there must exist a corner or else Γ is one of the curves of the arrangement. Since $K \subset I(G,H)$, it would follow that curves Γ and G do not intersect, a contradiction. Second note that if there is only one corner, then Γ is part of a self-intersecting curve of the arrangement, again a contradiction. Thus we can assume K has two corners. Since K is assumed not to have two convex corners, at least one of these, say x, is concave. Now suppose x is not of dual type. Let w and y (not necessarily distinct) be corners of K adjacent to x which are encountered by proceeding in the two possible directions around Γ from x. If $w = y$ and this corner is convex then Γ is composed of parts of two curves which intersect each other more than twice. If $w = y$ and this corner is concave, then Γ is composed of parts of two curves which either intersect each other more than twice or do not intersect the curve G. Thus w and y are actually distinct. Since K does not have two convex corners, w and y are not both convex and one of them, say y is concave. Let γ be the geodesic arc connecting x and y, and let a and b be the edges of γ touching x and y respectively. Since we assumed that x is not of dual type, Lemma 10 implies that $A(x,a)$ dives. Now $A(y,b)$ must surface or we would contradict IV. Consequently, applying Lemma 10 to the vertex y, we deduce that y is of dual type. This proves the lemma.

Let x and y be concave corners of dual type with surfacing arcs $A(x,a)$ and $A(y,b)$ respectively. Then x and y are said to be *consistent* if the vertices $w(a,x)$ and $w(y,b)$ separate x from y on the circuit Γ.

Now let $x_0, x_1, \cdots, x_s$ be a maximal sequence of concave vertices which are all of dual type and mutually consistent and which are such that each x_i is adjacent to x_{i+1} (provided $s > 0$). Define y and z to be adjacent corners to x_0 and x_s respectively and such that $y, x_0, \cdots, x_s, z$ are consecutive in that order in a uniform direction around Γ. Let e_0 be the geodesic arc connecting y to x_0 and let e_{s+1} be the geodesic arc connecting x_s to z. If $s > 0$, define e_i as the geodesic arc connecting

x_{i-1} to x_i for $i = 1, 2, \cdots, s$. Note that y and z may be identical with x_s and x_0 respectively, or y and z may be identical; we shall, however, shortly rule out these possibilities. Because of the consistency of the x_i, we may assume with no loss of generality that $A(x_i, e_i)$ dives while $A(x_i, e_{i+1})$ surfaces for $i = 0, 1, \cdots, s$. See Figure 3.

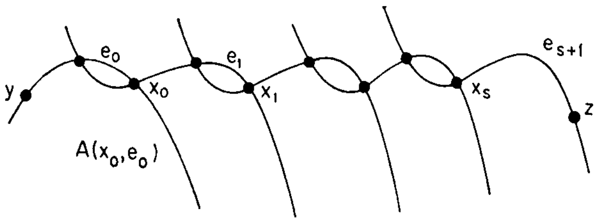

Fig. 3

LEMMA 12. *If* $s > 0$, $A(x_i, e_i) \cap A(x_{i+1}, e_{i+1}) = 0$, *for* $i = 0, 1, \cdots, s-1$.

PROOF. If the two arcs were to intersect, then $G(e_i)$ and $G(e_{i+1})$ would intersect at least three times.

Now $A(x_i, e_i)$ divides Γ into two arcs R_i and L_i, one of which contains e_i and the other of which contains e_{i+1}, for $i = 0, 1, \cdots, s$. Choose the notation so that $e_i \in L_i$ while $e_{i+1} \in R_i$. As a simple consequence of Lemma 12 we have the following lemma.

LEMMA 13.

$$L_0 \subset L_1 \subset \cdots \subset L_s \text{ and}$$

$$R_0 \supset R_1 \supset \cdots \supset R_s.$$

COROLLARY 14. $y \in L_0$ *and* $z \in R_0$; *furthermore* $y \neq z$, x_i, *and* $z \neq x_i$.

PROOF. That $y \in L_0$ and $z \in R_0$ follows immediately from Lemma 13 and the fact that $z \in R_s$. For the remainder, it is only necessary to show that neither y nor z belongs to $L_0 \cap R_0$. If $y \in R_0$ then $y = w(x_0, e_0)$ and $G(e_0)$ crosses itself at y, a contradiction, or $e_0 \cup A(x_0, e_0)$ is a curve of the arrangement. In the latter case this curve would not cross G since $K \subset I(G, H)$, and this is a contradiction. As regards z, because $A(x_s, e_s)$ dives, z is in the relative interior of the arc R_s and thus in the relative interior of R_0. Consequently it cannot lie in L_0.

Since $z \neq x_0$ and since $x_0, x_1, \cdots, x_s$ is a maximal consistent sequence of distinct dual type concave vertices, z must be convex or not of dual type or it must be of dual type but inconsistent with x_s. The latter two cases violate IV because of Lemma 10. Therefore neither case can occur and z must be convex. Since we are assuming K has no more than one convex corner, and since $y \neq z$, the corner y, which we henceforth denote y_0, must be concave and either not of dual type or of dual type but not consistent with x_0. In either case, $A(y_0, e_0)$ dives. Let y_1 be the corner of K other than x_0 which is adjacent to y_0. Let the diving arc $A(y_0, e_0)$ divide Γ into arcs L'_0 and R'_0 where $y_1 \in L'_0$ and $x_0 \in R'_0$. Since $G(e_0)$ does not cross itself, $L'_0 \subset L_0$. If K does not have two convex corners, then y_1 is concave. (Note that $y_1 \neq z$ since $y_1 \in L'_0 \subset L_0$.) By applying Lemma 10 and IV we see that y_1 is of dual type but inconsistent with x_0.

Let $y_1, y_2, \cdots, y_t$ be a maximal consistent sequence of concave vertices of dual type. Just as with the x_i we can define L'_i and R'_i at y_i and we have

$$L_0 \supset L'_0 \supset L'_1 \supset \cdots \supset L'_t.$$

Now if w is adjacent to y_t but different from y_{t-1} it follows that $w \in L'_t \subset L_0$, hence $w \neq z$. Now by the same type of analysis as that applied to z we show that w is a convex corner. Consequently K has at least two convex corners and we have therefore shown that all simply connected complexes contained in $I(G, H)$ have at least two convex corners. This completes the proof of Lemma 3.

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Department of Mathematics
Adelphi University
Garden City, Long Island, New York, U. S. A.